

COMBINATORICS OF THE z^{AB} THEOREM

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We give a combinatorial proof that the coefficient of z^{AB} in a certain rational function is a sum of two binomial coefficients.

1. Introduction

The q -Dyson conjecture was a constant term problem which was finally solved combinatorially by Zeilberger and Bressoud [10]. In this paper we shall give a combinatorial proof of another such theorem [4, Corollary 3].

Theorem 1. *Let A and B be positive integers. The coefficient of z^{AB} in*

$$\frac{(1 - \lambda\mu z^{A+B})^{A+B}}{(1 - \lambda z^A)^A (1 - \mu z^B)^B} \quad (1)$$

is

$$\binom{A+B-1}{B} \lambda^B + \binom{A+B-1}{A} \mu^A. \quad (2)$$

The analytic proof of Theorem 1 in [4] uses a special evaluation of a generalized hypergeometric series. Because of the extensive work on the combinatorics of these series, one might think that a combinatorial proof of Theorem 1 is routine. However, this is not true. There are two bijective models for such series. The first allows arbitrary parameters in the series and computes generating functions of objects [5, 7]; while the second restricts the parameters to be integers and counts objects in specific sets. It is theoretically possible to use the first model to derive results in the second model, although the involution principle may be necessary. The first model does not explain special conditions on the parameters, which do exist for Theorem 1. The relevant versions of the second model for Theorem 1 involve integer parameters which give positive terms in the series, whereas we need parameters which make the series alternate (see

Lemma 4). So Theorem 1 poses a much more difficult combinatorial problem.

We will in fact give a sign-reversing involution which proves a q -analogue (Theorem 6) of Theorem 1. Unfortunately, Theorem 6 has a finite sum which replaces (2), and thus is not as elegant as Theorem 1. Our involution will indicate why the q -analogue is more complicated.

The rest of the paper is organized in the following way. In Section 2 the combinatorial model for Theorem 1 and its q -analogue is given. The sign-reversing involution on ordered triples of partitions is given in Section 3. The final result is given in Section 4, along with a mischievous open bijection for binomial coefficients. Finally, remarks are made in Section 5.

2. Combinatorics of Theorem 1

If we put $x = \lambda z^A$ and $y = \mu z^B$ it is natural to consider

$$F(x, y) = \frac{(1 - xy)^{A+B}}{(1 - x)^A(1 - y)^B}. \quad (3)$$

Clearly $F(x, y)$ is the generating function for ordered triples (a, b, c) of

- (a) multisets a from an alphabet \mathcal{A} with A elements,
- (b) multisets b from an alphabet \mathcal{B} with B elements,
- (c) subsets c from an alphabet \mathcal{C} with $A + B$ elements.

The weights are defined by $w(\alpha) = x$, $w(\beta) = y$, and $w(\gamma) = -xy$, for $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$, and $\gamma \in \mathcal{C}$.

To construct a term in $F(\lambda z^A, \mu z^B)$ contributing to the coefficient of z^{AB} , we choose a multiset a with n_a elements from \mathcal{A} , a multiset b with n_b elements from \mathcal{B} , and a subset c with n_c elements from \mathcal{C} . We must have

$$(n_a + n_c)A + (n_b + n_c)B = AB. \quad (4)$$

This will give us a term with

$$\lambda^{n_a + n_c} \mu^{n_b + n_c}$$

in it. For such n_a , n_b , and n_c , the number of such set and multiset triples is

$$\binom{n_a + A - 1}{n_a} \binom{n_b + B - 1}{n_b} \binom{A + B}{n_c} \quad (5)$$

and the sign will be $(-1)^{n_c}$.

If we restrict $n_c = 0$, the solutions to (4) will have the following form. Let $r = \gcd(A, B)$. Write $A = ar$ and $B = br$. Then (4) implies $n_a = db$ and $n_b = (r - d)a$, $0 \leq d \leq r$.

In general, $n_a + n_c = db$ and $n_b + n_c = (r - d)a$. Each d will give a different power of λ and μ . For $d = r$, $n_b + n_c = 0$ implies $n_b = 0$ and $n_c = 0$ so that $n_a = rb = B$. For $d = 0$, $n_a + n_c = 0$ implies $n_a = 0$ and $n_c = 0$ so that $n_b = ra = A$.

The former gives the term

$$\binom{A+B-1}{B} \lambda^B$$

while the latter gives the term

$$\binom{A+B-1}{A} \mu^A.$$

These are precisely the two terms in Theorem 1.

For other values d , n_c may range between 0 and $\min\{db, (r-d)a\}$; each choice will give $\lambda^{db} \mu^{(r-d)a}$. Letting $c = r - d$ and $k = n_c$, and summing the terms in (5), Theorem 1 will follow from the following lemma.

Lemma 2. For $a, b, c, d > 0$,

$$\sum_k (-1)^k \binom{ac-1+b(c+d)-k}{ac-k} \binom{bd-1+a(c+d)-k}{bd-k} \binom{(a+b)(c+d)}{k} = 0. \quad (6)$$

Lemma 2 is a special case of a 2-balanced ${}_3F_2$ evaluation (see [4]). We begin by using Pascal's triangle for the third binomial coefficient in (6) to find

$$\begin{aligned} \sum_k (-1)^k \binom{ac-1+b(c+d)-k}{ac-k} \binom{bd-1+a(c+d)-k}{bd-k} \binom{(a+b)(c+d)-1}{k} \\ = \sum_k (-1)^k \binom{ac-2+b(c+d)-k}{ac-1-k} \binom{bd-2+a(c+d)-k}{bd-1-k} \binom{(a+b)(c+d)-1}{k}. \end{aligned} \quad (7)$$

To prove (7) we will evaluate both sides, and show that they are identical. This is the content of the following lemma, which is an alternating sign version of Saalschütz's ${}_3F_2$ evaluation.

Lemma 3. For $0 < u < B$ and $0 < v < A$,

$$\begin{aligned} \sum_k (-1)^k \binom{A-1+u-k}{u-k} \binom{B-1+v-k}{v-k} \binom{A+B-1}{k} \\ = \binom{A-1+u-v}{u} \binom{B-1+v-u}{v}. \end{aligned} \quad (8)$$

The q -analogue of Lemma 3 which we shall prove is Lemma 4.

Lemma 4. For $0 < u < B$ and $0 < v < A$,

$$\begin{aligned} \sum_k (-1)^k \begin{bmatrix} A-1+u-k \\ u-k \end{bmatrix}_q \begin{bmatrix} B-1+v-k \\ v-k \end{bmatrix}_q \begin{bmatrix} A+B-1 \\ k \end{bmatrix}_q q^{\binom{k}{2}} \\ = q^{uv} \begin{bmatrix} A-1+u-v \\ u \end{bmatrix}_q \begin{bmatrix} B-1+v-u \\ v \end{bmatrix}_q. \end{aligned} \quad (9)$$

We shall prove q -Saalschütz (Lemma 4) by interpreting the q -binomial coefficients as the generating functions for partitions which lie inside rectangles [1]. The $q = 1$ case (Lemma 3) will follow by interpreting the lattice paths of the Ferrers diagrams of the partitions as subsets.

3. The q -Saalschütz theorem

First we need some notation for sets of partitions. For any integer partition λ , let $w(\lambda) = q^{|\lambda|}$, where $|\lambda|$ denotes the number partitioned by λ . Let $\mathbf{PD}_k^{a,b}$ denote all partitions with k distinct parts, whose part sizes lie between a and b (inclusive). Similarly let $\mathbf{P}_k^{a,b}$ denote all partitions with k parts, whose part sizes lie between a and b (inclusive). If $\lambda \in \mathbf{PD}_k^{a,b}$, let $\text{sign}(\lambda) = (-1)^k$.

The left side of (9) is the generating function for the set S ,

$$S = \bigcup_k S_k = \bigcup_k \mathbf{PD}_k^{0,A+B-2} \times \mathbf{P}_{u-k}^{0,A-1} \times \mathbf{P}_{v-k}^{0,B-1}.$$

The right side of (9) is the generating function for T ,

$$T = \{u^v\} \times \mathbf{P}_u^{0,A-1-v} \times \mathbf{P}_v^{0,B-1-u},$$

where u^v is the partition with v parts of size u . Lemma 4 clearly now has this combinatorial interpretation.

Lemma 5. *For $0 < u < B$ and $0 < v < A$, there is a sign-reversing, weight-preserving involution Ψ on S with fixed point set Ψ_0 . Furthermore, all elements of Ψ_0 have positive sign and there is a weight-preserving bijection between Ψ_0 and T .*

Proof. First of all, we will show that $\Psi_0 = \{(\emptyset, \mu, \nu)\}$ where \emptyset is the empty partition ($k = 0$), $\mu \in \mathbf{P}_u^{0,A-1-v}$ and $\nu \in \mathbf{P}_v^{0,B-1}$. Since there is an obvious identification between $\mathbf{P}_v^{0,B-1}$ and $\{u^v\} \times \mathbf{P}_v^{0,B-1-u}$, the bijection between Ψ_0 and T is immediate.

We will define two separate involutions Ψ^* and $\tilde{\Psi}$. The involution Ψ^* will be defined on the set $S^* \subseteq S$ and the involution $\tilde{\Psi}$ will be defined on the set $\tilde{S} = S - S^*$. The involution Ψ^* will have no fixed points. The fixed points of $\tilde{\Psi}$ will be exactly $\{(\emptyset, \mu, \nu)\}$ described in the previous paragraph. Both $\tilde{\Psi}$ and Ψ^* will be based on the same involution, which we call Φ . It is also defined on the set S .

Let $(\lambda, \mu, \nu) \in S_k$. Write $\lambda = (\lambda_1, \dots, \lambda_k)$, $\mu = (\mu_1, \dots, \mu_{u-k})$ and $\nu = (\nu_1, \dots, \nu_{v-k})$. For degenerate cases, we assume $\lambda_0 = A + B - 1$, $\mu_0 = A - 1$ and $\nu_0 = B - 1$. We make the following definitions:

$$\begin{aligned} x &= \lambda_k \\ y &= \nu_{v-k} \\ z &= \mu_{u-k-y}, \quad \text{if } y \leq u - k \\ r &= \min\{t : x \leq \mu_{u-k-t} + t, t = 0, \dots, u - k\}, \quad \text{if } x \leq u - k + A - 1 \\ s &= x - r, \quad \text{if } x \leq u - k + A - 1. \end{aligned}$$

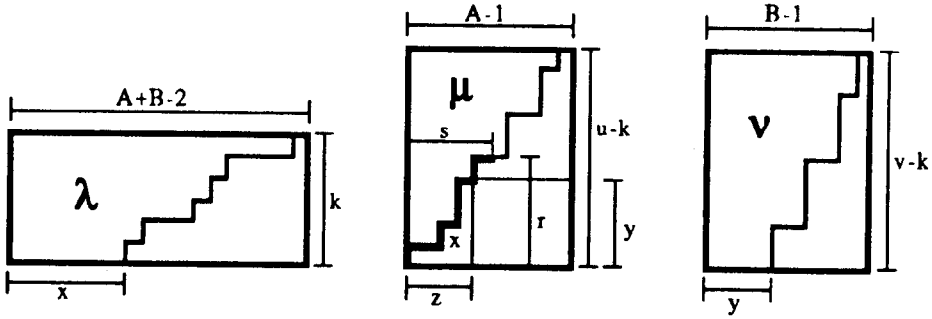


Fig. 1.

See Fig. 1. We can think of laying x along the boundary of the Ferrers diagram of μ . If x lies inside the $(A-1) \times (u-k)$ rectangle containing μ , then r and s are the height and width of x respectively.

The definition of Φ breaks into two cases. We let $\Phi(\lambda, \mu, \nu) = (\lambda', \mu', \nu')$. Associated with (λ', μ', ν') we have x', y', z', r', s' and k' .

Case 1. $x \leq y + z$ or both $y > u - k$ and $x \leq u - k + A - 1$. Note that in this case r and s must be defined and, in fact, $r \leq y$. In this case, define (λ', μ', ν') as follows:

$$\begin{aligned} k' &= k - 1, \\ \lambda' &= (\lambda_1, \dots, \lambda_{k-1}), \\ \mu' &= (\mu_1, \dots, \mu_{u-k-r}, s, \mu_{u-k-r+1}, \dots, \mu_{u-k}), \quad \text{and} \\ \nu' &= (\nu_1, \dots, \nu_{v-k}, r). \end{aligned}$$

We then have

$$\begin{aligned} x' &> x, \\ y' &= r, \\ z' &= s, \\ r' &> r, \quad \text{if } x' \leq u - k' + A - 1, \quad \text{and} \\ s' &\geq s, \quad \text{if } x' \leq u - k' + A - 1. \end{aligned}$$

Case 2. $x > y + z$. Note that in this case z must be defined and either r is not defined or $r > y$. Define (λ', μ', ν') as follows:

$$\begin{aligned} k' &= k + 1, \\ \lambda' &= (\lambda_1, \dots, \lambda_k, y + z), \\ \mu' &= (\mu_1, \dots, \mu_{u-k-y-1}, \mu_{u-k-y+1}, \dots, \mu_{u-k}), \quad \text{and} \\ \nu' &= (\nu_1, \dots, \nu_{v-k-1}). \end{aligned}$$

We then have

$$\begin{aligned} x' &= y + z, \\ y' &= r \geq y, \\ z' &\geq z, \quad \text{if } y' \leq u - k', \\ r' &= y, \quad \text{and} \\ s' &= z. \end{aligned}$$

Note that if (λ, μ, ν) is in Case 1, then (λ', μ', ν') will be in Case 2 and vice versa. Also Φ is sign-reversing and weight-preserving. The fixed points occur exactly when none of z , r , or s are defined. This will happen when $y > u - k$ and $x > u - k + A - 1$.

Unfortunately, Φ does not give us a proof of Lemma 5—its fixed point set is too large. So we must restrict Φ to a smaller subset S^* of S and look for another involution on the complement. It turns out that the new involution will also be defined from Φ . This brings us to the definition of S^* and \bar{S} . Let

$$w = \min\{t : \mu_{u-k-t} \geq A - v\}.$$

Then

$$S^* = \{(\lambda, \mu, \nu) : x < A - v + w \text{ or } y < w\},$$

$$\bar{S} = \{(\lambda, \mu, \nu) : x \geq A - v + w \text{ and } y \geq w\}.$$

Now define $\Psi^* = \Phi|S^*$. We must verify that Ψ^* is well-defined on S^* and has no fixed points.

Let $(\lambda, \mu, \nu) \in S^*$ and let $\Phi(\lambda, \mu, \nu) = (\lambda', \mu', \nu')$ with the same definitions of $x, y, z, r, s, x', y', z', r'$ and s' as before. Also let w be defined as above and let w' be the corresponding value for (λ', μ', ν') .

Suppose (λ, μ, ν) is in Case 1. It is clear from Fig. 2 that $r \leq w$ and $w' = w + 1$. Thus $y' = r \leq w < w'$ so that (λ', μ', ν') is in S^* .

Suppose (λ, μ, ν) is in Case 2. It is clear from Fig. 3 that $w' = w - 1$ and that if $y < w$ then $z < A - v$. Thus, either $x' = y + z < x < A - v + w$, or $x' = y + z < w + z < A - v + w$. In either case $x' < A - v + w - 1 = A - v + w'$, and again (λ', μ', ν') is in S^* .

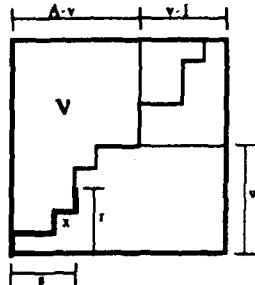


Fig. 2.

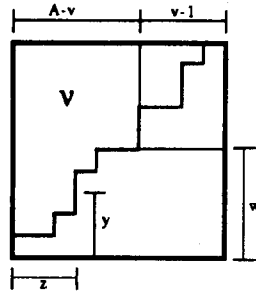


Fig. 3.

Since either $y < w \leq u - k$ or $x < A - v + w \leq A - 1 + w \leq A - 1 + u - k$, Ψ^* has no fixed points.

We now turn our attention to \tilde{S} and the definition of $\tilde{\Psi}$. Suppose $(\lambda, \mu, \nu) \in \tilde{S}$. Define

$$\Gamma(\lambda, \mu, \nu) = ((A - v)^{u-k-w}, (A - v + w)^k, w^{v-k}, \tau, \tilde{\lambda}, \tilde{\mu}, \tilde{\nu})$$

where

$$\tau \in P_w^{0, A-v-1},$$

$$\tilde{\lambda} \in PD_k^{0, B+v-w-2} = PD_k^{0, \tilde{A}+\tilde{B}-2},$$

$$\tilde{\mu} \in P_{v-k}^{0, B-1-w} = P_{\tilde{u}-k}^{0, \tilde{A}-1},$$

$$\tilde{\nu} \in P_{u-k-w}^{0, v-1} = P_{\tilde{v}-k}^{0, \tilde{B}-1}$$

and

$$\tilde{A} = B - w,$$

$$\tilde{B} = v,$$

$$\tilde{u} = v,$$

$$\tilde{v} = u - w.$$

See Fig. 4.

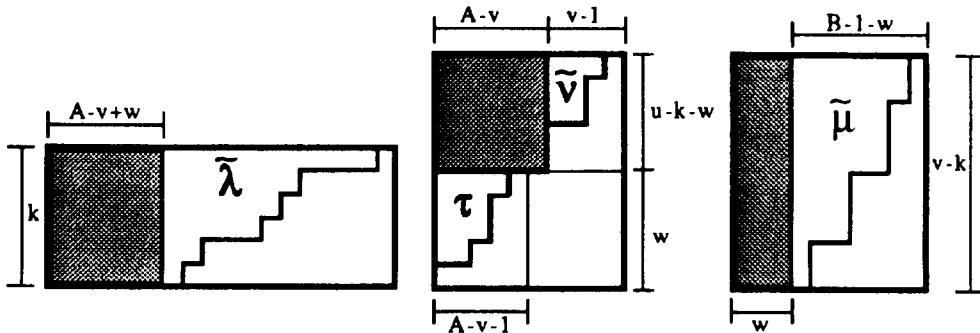


Fig. 4.

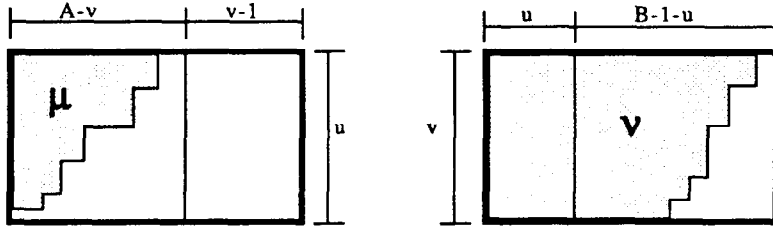


Fig. 5.

If $\tilde{v} = u - w > 0$, we can apply the involution Φ to the triple $(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu})$ to get $(\tilde{\lambda}', \tilde{\mu}', \tilde{\nu}')$. We can write $((A - v)^{u-k-w}, (A - v + w)^k, w^{v-k}, \tau)$ as $((A - v)^{u-k'-w}, (A - v + w)^{k'}, w^{v-k'}, \tau)$ where $k' = k \pm 1$. We may then piece the partitions back together by observing that there is an element $(\lambda', \mu', \nu') \in S_k$ such that

$$\Gamma(\lambda', \mu', \nu') = ((A - v)^{u-k'-w}, (A - v + w)^{k'}, w^{v-k'}, \tau, \tilde{\lambda}', \tilde{\mu}', \tilde{\nu}').$$

The fixed points of the involution Φ applied to $(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu})$ would occur when $\tilde{x} > \tilde{u} - k + \tilde{A} - 1 = v - k + B - w - 1$. Since \tilde{x} is the smallest part of $\tilde{\lambda}$ and $\tilde{\lambda} \in \mathbf{PD}_k^{0, \tilde{A} + \tilde{B} - 2}$, $\tilde{x} \leq \tilde{A} + \tilde{B} - 2 - k + 1 = v - k + B - w - 1$. Therefore, Φ in this case has no fixed points.

Thus, when $w < u$, we let $\tilde{\Psi} = \Gamma^{-1} \circ \Phi \circ \Gamma$.

If $w = u$, then $k = 0$, $\lambda = \emptyset$, $\mu = \tau \in \mathbf{P}_u^{0, A - u - 1}$ and $\nu \in \mathbf{P}_v^{u, B - 1}$. See Fig. 5. These are exactly the fixed points we were seeking. \square

4. The final identity

If we apply Lemma 3 to both sides of (7) (with $A = b(c + d)$, $B = a(c + d)$, $u = ac$, $v = bd$; and $A = b(c + d)$, $B = a(c + d)$, $u = ac - 1$, $v = bd - 1$) we must show

$$\binom{c(a+b)-1}{ac} \binom{d(a+b)-1}{bd} = \binom{c(a+b)-1}{bc} \binom{d(a+b)-1}{ad}. \quad (10)$$

We could not give a simple bijection which proved (10). If one allows multiplication of (10) by $abcd$, then the fact that $(ac)(bd) = (bc)(ad)$ gives an easy combinatorial proof.

For the q -analogue of Theorem 1, our involution proves the following theorem. We use the notation

$$(x)_A = \prod_{i=0}^{A-1} (1 - xq^i).$$

Theorem 6. Let A and B be positive integers, $r = \gcd(A, B)$, $A = ar$, and $B = br$. The coefficient of z^{AB} in

$$\frac{(\lambda \mu z^{AB})_{A+B}}{(\lambda z^A)_A (\mu z^B)_B} \quad (11)$$

is

$$\begin{bmatrix} A+B-1 \\ B \end{bmatrix}_q \lambda^B + \begin{bmatrix} A+B-1 \\ A \end{bmatrix}_q \mu^A + \sum_{d=1}^{r-1} \lambda^{bd} \mu^{ac} q^{abcd} K_d,$$

where $c = r - d$ and

$$K_d = \begin{bmatrix} c(a+b)-1 \\ ac \end{bmatrix}_q \begin{bmatrix} d(a+b)-1 \\ bd \end{bmatrix}_q - q^{ad+bc} \begin{bmatrix} c(a+b)-1 \\ ac-1 \end{bmatrix}_q \begin{bmatrix} d(a+b)-1 \\ bd-1 \end{bmatrix}_q.$$

This may explain why a bijection for (10) is not transparent.

5. Further comments

Many combinatorial proofs of Saalschütz's ${}_3F_2$ evaluation and its q -analogue have appeared. Two generating function proofs are given in [5] and [7]. To translate these to a proof of Lemma 2 would use the involution principle. The other bijective proofs do not have signed sets [2, 3, 6, 9]. For example, if A and B are negative integers, and the signs of x and y are changed, an expansion for (3) equivalent to an unsigned version Saalschütz's theorem can be done combinatorially [8].

Several of the results in [4] have combinatorial proofs which follow directly from the constructions in this paper. These include Theorem 10, Corollary 11, and (6.2).

The involution Φ defined in Section 3 can be used to prove directly the $A \rightarrow \infty$, $B \rightarrow \infty$ limiting case of Lemma 4.

A multivariable version of Theorem 1 is still unknown.

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